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A STUDY ON THE HYPERBOLA $3x^2+7xy+3y^2-13x-13y+9=0$

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ABSTRACT

This paper aims at obtaining non-zero distinct integer solutions to the hyperbola represented by the binary quadratic equation $3x^2+7xy+3y^2-13x-13y+9=0$. A few interesting relations among the solutions are presented. Employing the solutions of the given hyperbola, solutions for other choices of hyperbolas and parabolas are presented.

Keywords: *Binary quadratic, non-homogeneous quadratic, hyperbola, Integer solutions.*
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I. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-18], the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $3x^2+7xy+3y^2-13x-13y+9=0$.The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

II. METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$3x^2+7xy+3y^2-13x-13y+9=0 \quad (1)$$

Introduction of the transformations

$$x=X+1, y=Y+1 \quad (2)$$

in (1) gives

$$3X^2+3Y^2+7YX-4=0 \quad (3)$$

Substitution of the transformations,

$$X=u+v,\ Y=u-v,\ u\neq v\neq 0 \quad (4)$$

in (3) leads to

$$v^2=13u^2-4 \quad (5)$$

whose smallest positive integer solution is $u_0=1, v_0=3$

Now, consider the pellian equation

$$v^2=13u^2+1 \quad (6)$$

whose general solution $(\tilde{u}_n,\tilde{v}_n)$ is given by

$$\tilde{v}_n=\frac{1}{2}f_n\ ,\ \tilde{u}_n=\frac{1}{2\sqrt{13}}g_n$$

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where $g_n = [(649+180\sqrt{13})^{n+1} - (649-180\sqrt{13})^{n+1}]$

$$f_n = [(649+180\sqrt{13})^{n+1} + (649-180\sqrt{13})^{n+1}], \quad n = 0,1,2,3........$$

Applying Brahmagupta lemma between the solutions (u_0, v_0) and $(\tilde{u}_n, \tilde{v}_n)$, we have

$$u_{n+1} = \frac{f_n}{2} + \frac{3g_n}{2\sqrt{13}}$$

$$v_{n+1} = \frac{3f_n}{2} + \frac{13g_n}{2\sqrt{13}}$$

Taking the advantage of (2) and (4), the sequence of integral solutions of (1) can be written as

$$13x_{n+1} = 26f_n + 8\sqrt{13}g_n + 13 \quad (7)$$

$$13y_{n+1} = -(13f_n + 5\sqrt{13}g_n) + 13 \quad (8)$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1). The above values of x_n and y_n satisfy respectively the following recurrence relations.

$$x_{n+3} = 1298x_{n+2} - x_{n+1} - 1296$$

$$y_{n+3} = 1298y_{n+2} - y_{n+1} - 1296, \quad n = 0,1,2,.........$$

Some numerical examples of x and y satisfying (1) are given in the Table :1 below.

Table 1: Examples

n	x_{n+1}	y_{n+1}
-1	5	-1
0	5477	-3097
1	7107845	-4021201
2	9225976037	-5219517097

From the above table, we observe some interesting relations among the solutions which are presented below.

1. x_{n+1} values are positive and y_{n+1} values are negative. Both values are odd.
2. Each of the following expressions is a Nasty number :

- $\frac{1}{45}(-1234x_{2n+2} + x_{2n+3} + 1773)$
- $\frac{1}{58410}(-1601731x_{2n+2} + x_{2n+4} + 2302650)$
- $3(5x_{2n+2} + 8y_{2n+2} - 9)$
- $\frac{3}{611}(-5585x_{2n+2} - 8y_{2n+3} + 8037)$
- $\frac{3}{793079}(-7249325x_{2n+2} - 8y_{2n+4} + 10421649)$

3. Each of the following expressions is a Cubical integer:

- $\frac{1}{270}(-1234x_{3n+3} + x_{3n+4} - 3702x_{n+1} + 3x_{n+2} + 4932)$
- $\frac{1}{350460}(-1601731x_{3n+3} + x_{3n+5} - 4805193x_{n+1} + 3x_{n+3} + 6406920)$
- $\frac{1}{2}(5x_{3n+3} + 8y_{3n+3} + 15x_{n+1} + 24y_{n+1} - 52)$

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- $\frac{1}{1222}(-5585x_{3n+3}-8y_{3n+4}-16755x_{n+1}-24y_{n+2}+22372)$
- $\frac{1}{1586158}(-7249325x_{3n+3}-8y_{3n+5}-21747975x_{n+1}-24y_{n+3}+28997332)$

4. Each of the following expressions is a Bi-Quadratic integer:

- $\frac{1}{270}(-1234x_{4n+4}+x_{4n+5}-4936x_{2n+2}+4x_{2n+3}+7785)$
- $\frac{1}{350460}(-1601731x_{4n+4}+x_{4n+6}-6406924x_{2n+2}+4x_{2n+4}+10111410)$
- $\frac{1}{2}(5x_{4n+4}+8y_{4n+4}+20x_{2n+2}+32y_{2n+2}-53)$
- $\frac{1}{1222}(-5585x_{4n+4}-8y_{4n+5}-22340x_{2n+2}-32y_{2n+3}+35297)$
- $\frac{1}{1586158}(-7249325x_{4n+4}-8y_{4n+6}-28997300x_{2n+2}-32y_{2n+4}+45763613)$

5. Each of the following expressions is a Quintic integer:

- $\frac{1}{270}(-1234x_{5n+5}+x_{5n+6}-6170x_{3n+3}+5x_{3n+4}-12340x_{n+1}+10x_{n+2}+19728)$
- $\frac{1}{350460}\begin{pmatrix}-1601731x_{5n+5}+x_{5n+7}-8008655x_{3n+3}+5x_{3n+5}-16017310x_{n+1}\\+10x_{n+3}+25627680\end{pmatrix}$
- $\frac{1}{2}(5x_{5n+5}+8y_{5n+5}+25x_{3n+3}+40y_{3n+3}+50x_{n+1}+80y_{n+1}-208)$
- $\frac{1}{1222}(-5585x_{5n+5}-8y_{5n+6}-27925x_{3n+3}-40y_{3n+4}-55850x_{n+1}-80y_{n+2}+89488)$
- $\frac{1}{1586158}\begin{pmatrix}-7249325x_{5n+5}-8y_{5n+7}-36246625x_{3n+3}-40y_{3n+5}\\-72493250x_{n+1}-80y_{n+3}+188482658\end{pmatrix}$

6. Relations among the solutions:

- $13x_{n+3}=-13x_{n+1}+16874x_{n+2}-16848$
- $1298x_{n+2}=x_{n+1}+x_{n+3}+1296$
- $319337750160y_{n+3}=-228000x_{n+1}-180662603121x_{n+3}+500000581160$
- $1222y_{n+3}=2160x_{n+1}+1586158y_{n+2}-1587096$
- $793079x_{n+3}=-x_{n+1}-1401840y_{n+3}+2194920$
- $793079y_{n+2}=-1080x_{n+1}+611y_{n+3}+793548$

Remarkable Observations

(i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below.

Table 2: Hyperbolas

S.NO	Hyperbolas	(X,Y)
1	$208X^2-Y^2=60652800$	$\begin{pmatrix}-1234x_{n+1}+x_{n+2}+1233,\\17797x_{n+1}-13x_{n+2}-17784\end{pmatrix}$
2	$208X^2-Y^2=102188080051200$	$\begin{pmatrix}-1601731x_{n+1}+x_{n+3}+1601730,\\23100493x_{n+1}-13x_{n+3}-23100480\end{pmatrix}$

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3	$13X^2 - Y^2 = 208$	$\begin{pmatrix} 5x_{n+1} + 8y_{n+1} - 13, \\ -13x_{n+1} - 26y_{n+1} + 39 \end{pmatrix}$
4	$X^2 - 13Y^2 = 5973136$	$\begin{pmatrix} -5585x_{n+1} - 8y_{n+2} + 5593, \\ 1549x_{n+1} + 2y_{n+2} - 1551 \end{pmatrix}$
5	$13X^2 - Y^2 = 130826654450128$	$\begin{pmatrix} -7249325x_{n+1} - 8y_{n+3} + 7249333, \\ 26137813x_{n+1} + 26y_{n+3} - 26137839 \end{pmatrix}$

(ii) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below.

Table3: Parabolas

S.NO	Parabolas	(X,Y)
1	$56160X - Y^2 = 60652800$	$\begin{pmatrix} -1234x_{2n+2} + x_{2n+3} + 1773, \\ 17797x_{n+1} - 13x_{n+2} - 17784 \end{pmatrix}$
2	$72895680X - Y^2 = 102188080051200$	$\begin{pmatrix} -1601731x_{2n+2} + x_{2n+4} + 2302650, \\ 23100493x_{n+1} - 13x_{n+3} - 23100480 \end{pmatrix}$
3	$26X - Y^2 = 208$	$\begin{pmatrix} 5x_{2n+2} + 8y_{2n+2} - 9, \\ -13x_{n+1} - 26y_{n+1} + 39 \end{pmatrix}$
4	$94X - Y^2 = 459472$	$\begin{pmatrix} -5585x_{2n+2} - 8y_{2n+3} + 8037, \\ 1549x_{n+1} + 2y_{n+2} - 1551 \end{pmatrix}$
5	$20620054X - Y^2 = 130826654450128$	$\begin{pmatrix} -7249325x_{2n+2} - 8y_{2n+4} + 10421649, \\ 26137813x_{n+1} + 26y_{n+3} - 26137839 \end{pmatrix}$

III. CONCLUSION

In conclusion, one may search for other patterns of solutions and their corresponding properties.

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